

Dilaton minimally coupled to $2 + 1$ Einstein–Maxwell fields; stationary cyclic symmetric black holes

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Using the Schwarzschild coordinate frame for a static cyclic symmetric metric in $2 + 1$ Einstein gravity coupled to a electric Maxwell field and a dilaton logarithmically depending on the radial coordinate in the presence of an exponential potential the general solution of the Einstein–Maxwell–dilaton equations is derived and it is identified with the Chan–Mann charged dilaton solution. Via a general $SL(2, R)$ –transformation, applied on the obtained charged dilaton metric, a family of stationary dilaton solutions has been generated; these solutions possess five parameters: dilaton and cosmological constants, charge, momentum, and mass for some values of them. All the exhibited solutions have been characterized by their quasi-local energy, mass, and momentum through their series expansions at spatial infinity. The structural functions determining these solutions increase as the radial coordinate does, hence they do not exhibit an dS–AdS behavior at infinity. Moreover, the algebraic structure of the Maxwell field, energy-momentum, and Cotton tensors is given explicitly.

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I. INTRODUCTION

During the last two decades three–dimensional gravity has received some attention, in particular, in topics such as: black hole physics, search of exact solutions, quantization of fields coupled to gravity, cosmology, topological aspects, and others. This interest in part has been motivated by the discovery, in 1992, of the $2 + 1$ stationary circularly symmetric Bañados–Teitelboim–Zanelli (BTZ) anti-de Sitter black hole [1], see also [2–4], which possesses certain features inherent to $3 + 1$ black holes. On the other hand, it is believed that $2 + 1$ gravity may provide new insights towards a better understanding of the physics of $3 + 1$ quantum gravity. The list of references on the mentioned–above topics is rather vast. In particular, on exact solutions in $2 + 1$ gravity one finds works on point masses, cosmological universes, perfect fluid solutions, dilaton fields, scalar fields minimally and non–minimally coupled to gravity, electromagnetic fields, among others. Nevertheless, the literature on stationary rotating scalar field solutions is rather scarce [6, 7], most of the known solutions of this class are static [8–10].

The purpose of this contribution is to determine families of static and stationary cyclic symmetric solutions of the $2 + 1$ gravity, in the Schwarzschild coordinate frame, for a minimally coupled dilaton field depending logarithmically on radial coordinate and allowing for an exponential potential in the presence of an electromagnetic Maxwell field. The static cyclic symmetric metric can always be described by three structural functions $ds^2 = -A(r)^2 dt^2 + B(r)^2 dr^2 + C(r)^2 dt^2$, leaving still a freedom in the choice of the r –coordinate, which can be used to fix the metric structure, for instance:

the Schwarzschild coordinate frame: $ds^2 = -A(r)^2 dt^2 + B(r)^2 dr^2 + r^2 dt^2$, or the $g_{tt} = -1/g_{rr}$ coordinate frame: $ds^2 = -F(r)^2 dt^2 + dr^2/F(r)^2 + H(r)^2 dt^2$. In one of the well-known works on dilaton [6], this last metric was used and it was also assumed $H(r) = r^{N/2}$ making troublesome the establishing of uniqueness of the derived solutions under certain conditions.

The Schwarzschild coordinate frame results adequate when searching for electric solutions within Maxwell electrodynamics, nevertheless it yields to troubles when dealing with magnetic fields. On these lines of thinking, integrating for Maxwell fields, Peldan [5] succeeded to integrate for the first time in the usual Schwarzschild gauge the electrostatic Einstein–Maxwell field equations with cosmological constant. Nevertheless, in the search for a magnetic solution, Peldan’s Section 5.1.2, one finds the following comment: “the reason why I did not choose the Schwarzschild gauge from the beginning, is that in choosing that gauge, I not have been able to find an explicit solution for the metric”. The integration was accomplished in the $g_{tt} = -1/g_{rr}$ –gauge, leaving the remaining metric function free.

This paper is organized as follows. Section II is devoted to the derivation of the general static cyclic symmetric solution for a charged dilaton field in the Schwarzschild coordinate frame, $g_{\theta\theta} = r^2$; this solution is equipped with four parameters: cosmological and dilaton constants, mass, and charge. It should be pointed out that this solution occurs to be equivalent to the charged static dilaton Chan-Mann solution [7], see Sec. III C. The evaluation of the quasi local mass is carried out. The classification of the Maxwell field and energy momentum tensor is accomplished, as well as the algebraic characterization of the Cotton tensor is done.

In Section III, using $SL(2, R)$ -transformations of the Killing coordinates, a family of rotating metrics coupled to a dilaton field is generated. A particular transformation gives rise to a charged generalization of the rotating Chan-Mann solution [7], see Sec. III C. For this class solutions the determination of the quasi local energy, mass, and momentum is carried out. This family of solution is endowed with five parameters; in particular they allow the interpretation of mass, angular momentum, charge, dilaton and cosmological constant. Moreover, the algebraic classification of the field, energy-momentum and Cotton tensors is accomplished. A summary of results is presented in Concluding Remarks IV.

A. Einstein-Maxwell-scalar field equations

The action to be considered in this work dealing (2+1)-dimensional gravity is given by

$$\mathcal{S} = \int d^3x \sqrt{-g} \left[R - \frac{B}{2} \nabla_\mu \Psi \nabla^\mu \Psi + 2e^{b\Psi} \Lambda - e^{-4a\Psi} F^2 \right], \quad (1.1)$$

where Λ, b are arbitrary at this stage parameters, Ψ is the massless minimally coupled scalar field, R is the scalar curvature, and $F^2 = F_{\mu\nu} F^{\mu\nu}$ the electromagnetic invariant. The variations of this action yield the dynamical equations

$$\begin{aligned} R_{\mu\nu} &= \frac{B}{2} \nabla_\mu \Psi \nabla_\nu \Psi - 2g_{\mu\nu} e^{b\Psi} \Lambda + 2e^{-4a\Psi} (F_\mu{}^\alpha F_{\nu\alpha} - g_{\mu\nu} F^2), \\ \frac{B}{2} \nabla^\mu \nabla_\mu \Psi + b e^{b\Psi} \Lambda + 2a e^{-4a\Psi} F^2 &= 0, \\ \nabla^\mu (e^{-4a\Psi} F_{\mu\nu}) &= 0. \end{aligned} \quad (1.2)$$

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II. GENERAL STATIC CYCLIC SYMMETRIC BLACK HOLE SOLUTION COUPLED TO A SCALAR FIELD $\Psi(r) = k \ln(r)$

The static cyclic symmetric metric in the 2 + 1 Schwarzschild coordinate frame is given by

$$g = -N(r)^2 dt^2 + \frac{dr^2}{L(r)^2} + r^2 d\phi^2. \quad (2.1)$$

The electromagnetic field equations for the tensor field $F_{\mu\nu} = 2F_{tr}\delta_{[\mu}^t\delta_{\nu]}^r$, and the dilaton $\Phi(r) = k \ln(r)$ becomes

$$EQ_F = \frac{d}{dr} \frac{F_{tr} L r^{-4ak+1}}{N} \rightarrow F_{tr} = Q \frac{N}{L} r^{4ak-1}. \quad (2.2)$$

The simplest Einstein equations occurs to be $R_{11} + R_{22} L^2 N^2$, which yields

$$\frac{1}{N} \frac{d}{dr} N - \frac{1}{L} \frac{d}{dr} L - \frac{1}{2} \frac{B k^2}{r} = 0, \quad (2.3)$$

thus one gets

$$N(r) = C_N L(r) r^{B k^2/2}. \quad (2.4)$$

On the other hand, the equation R_{33} gives a first order equation for $L^2 = Y(r)$, namely

$$\frac{d}{dr} Y(r) + \frac{1}{2} \frac{B k^2 Y(r)}{r} + 2 \frac{r^{4ak} Q^2}{r} - 2 \Lambda r^{bk+1} = 0 \quad (2.5)$$

integrating one obtains

$$L(r)^2 = Y(r) = -4 \frac{r^{4ak} Q^2}{B k^2 + 8ak} + 4 \frac{\Lambda r^{2+bk}}{4 + B k^2 + 2bk} + r^{-1/2 B k^2} C_1 \quad (2.6)$$

Substituting this expression of $Y(r)$ into the remaining scalar field equation

$$\frac{d}{dr} Y(r) + \frac{1}{2} \frac{B k^2 Y(r)}{r} - 8 \frac{r^{4ak} a Q^2}{B k r} + 2 \frac{b \Lambda r^{bk+1}}{B k} = 0, \quad (2.7)$$

one arrives at relationships between constants, namely

$$a = -\frac{1}{4 B k}, \quad b = -B k. \quad (2.8)$$

Therefore, the general charged dilaton static solution can be given as

$$\begin{aligned} g &= -C_N^2 r^{B k^2} L(r)^2 dt^2 + \frac{dr^2}{L(r)^2} + r^2 d\phi^2, \\ L(r)^2 &= \left(r^{B k^2/2} C_1 + 4 \frac{r^2 \Lambda}{4 - B k^2} + 4 \frac{Q^2}{B k^2} \right) r^{-B k^2}, \quad k^2 \neq \frac{4}{B}, \\ F_{\mu\nu} &= 2 F_{tr} \delta_{[\mu}^t \delta_{\nu]}^r, \quad F_{tr} = Q C_N r^{-1/2 B k^2 - 1} = -A_{t,r} \rightarrow A_t = 2 \frac{Q C_N}{B k^2} r^{-B k^2/2}, \\ \Psi(r) &= k \ln(r), \end{aligned} \quad (2.9)$$

endowed with four relevant parameters: in particular, one may identify the mass $M = -C_1$, cosmological constant $\Lambda \rightarrow \pm \frac{1}{l^2}$, dilaton parameter k , and the charge Q . The constant C_N can be absorbed by scaling the coordinate t , thus it can be equated to unit, $C_N \rightarrow 1$. Moreover, one has to set the charge Q to zero, $Q = 0$, when looking for the limiting solutions for vanishing dilaton $k = 0$, which are just the dS and AdS solutions with parameters $C_1 = \pm M$ respectively, and $C_N = 1$. There is no static electrically charged limit of this solution for vanishing dilaton field.

The constant Λ can be equated to minus the standard cosmological constant $\Lambda_s = \pm \frac{1}{l^2}$; indeed, by setting in (2.9)

$$\begin{aligned} \Lambda &= \pm \frac{1}{l^2} \alpha^2, \quad r \rightarrow r \alpha^{2/(B k^2)}, \quad \phi \rightarrow \phi \alpha^{-2/(B k^2)}, \quad Q \rightarrow Q \alpha^{(1+2/(B k^2))}, \\ C_1 &\rightarrow C_1 \alpha^{1+4/(B k^2)}, \quad C_N \rightarrow C_N \alpha^{-(1+2/(B k^2))}, \end{aligned} \quad (2.10)$$

one arrives at the metric (2.9) with $\Lambda = \pm \frac{1}{l^2}$. Notice that the Λ used by in Chan–Mann works, when considered as a cosmological constant, differs from the standard cosmological constant $\Lambda_s = \pm \frac{1}{l^2} = -\Lambda$, where $+$ and $-$ stand correspondingly for de Sitter and Anti de Sitter (AdS).

If $B > 0$, $\Lambda (4 - B k^2) > 0$, then one has

- a) dS horizonless: $\Lambda < 0 \wedge \{k < -\frac{\sqrt{B}}{\sqrt{2}}, k > \frac{\sqrt{B}}{\sqrt{2}}\}, C_1 > 0$,
- b) dS cosmological singularity: $\Lambda < 0 \wedge \{k < -\frac{\sqrt{B}}{\sqrt{2}}, k > \frac{\sqrt{B}}{\sqrt{2}}\}, C_1 < 0$,
- c) AdS horizonless: $\Lambda > 0 \wedge \{-\frac{\sqrt{B}}{\sqrt{2}} < k < \frac{\sqrt{B}}{\sqrt{2}}\}, C_1 > 0$,
- d) AdS black hole: $\Lambda > 0 \wedge \{-\frac{\sqrt{B}}{\sqrt{2}} < k < \frac{\sqrt{B}}{\sqrt{2}}\}, C_1 < 0$,

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- e) dS cosmological singularity: $\Lambda < 0 \wedge \{-\frac{\sqrt{B}}{\sqrt{2}} < k < \frac{\sqrt{B}}{\sqrt{2}}\}, C_1 > 0$,
- f) AdS event horizon: $\Lambda > 0 \wedge \{k < -\frac{\sqrt{B}}{\sqrt{2}}, k > \frac{\sqrt{B}}{\sqrt{2}}\}, C_1 > 0$.

The norm of the normal vector $n_\mu = \nabla_\mu r$ to the surface $r = r_0 = \text{const.}$ is given by $g^{rr}(r_0) = L(r_0)^2$. It becomes a null vector at the horizon r_h of the black hole solution with $B > 0$, $k^2 < 4/B$, $C_1 < 0$, and $\Lambda = \frac{1}{l^2}$ which is determined as the outer root of $g^{rr}(r) = L(r)^2 = 0$, namely

$$r^{B k^2/2} C_1 + 4 \frac{r^2 \Lambda}{4 - B k^2} + 4 \frac{Q^2}{B k^2} = 0. \quad (2.11)$$

The electromagnetic invariant of this solution amounts to

$$F_{\mu\nu} F^{\mu\nu} = -2 \frac{Q^2 C_1^2}{C_N^2} r^{-2(1+Bk^2)}.$$

Accomplishing in the general above solution (2.9) the r -transformation and constant parameterizations:

$$r \rightarrow r^{N/2}, \quad k = \pm \frac{\sqrt{2}}{\sqrt{B}} \sqrt{\frac{2-N}{N}},$$

one gets, modulo minor constants arrangements, the Chan–Mann static solution [6, 7]:

$$\begin{aligned} ds^2 &= -C_N^2 U(r) dt^2 + \frac{N^2}{4 U(r)} dr^2 + r^N d\phi^2, \\ U(r) &= C_1 r^{1-N/2} + \frac{2N\Lambda}{(3N-2)} r^N + 4 \frac{N}{2-N} Q^2, \\ F_{\mu\nu} &= 2F_{tr} \delta_{[\mu}^t \delta_{\nu]}^r, \quad F_{tr} = \frac{N}{2} Q C_N r^{N/2-2} = -A_{t,r} \rightarrow A_t = \frac{N}{2-N} \frac{Q C_N}{r^{N/2-1}}, \\ \Psi(r) &= \pm \sqrt{\frac{N(2-N)}{2B}} \ln(r), \quad 0 < N < 2, \quad N \neq 2/3. \end{aligned} \quad (2.12)$$

A. Quasi local momentum, energy and mass for the static charged black hole solution coupled to a scalar field

To characterize non asymptotically flat solutions one uses the Brown–York formalism [11] of quasi–local momentum, energy, and mass quantities, which for the stationary cyclic symmetric metric

$$g = -N(r)^2 dt^2 + \frac{dr^2}{L(r)^2} + K(r)^2 (d\phi + W(r)dt)^2. \quad (2.13)$$

are given by:

$$j_\phi = \frac{1}{2\pi} \frac{L}{N} K^2 W_{,r}|_{r \rightarrow \infty}, \quad J(\partial/\partial\phi) = \frac{L}{N} K^3 W_{,r}|_{r \rightarrow \infty}, \quad J = 2\pi K j_\phi, \quad (2.14)$$

$$\epsilon(\partial/\partial t) = -\frac{1}{\pi K} L K_{,r}|_{r \rightarrow \infty} - \epsilon_0, \quad E(\partial/\partial t) = -2 L K_{,r}|_{r \rightarrow \infty} - 2\pi \epsilon_0 K|_{r \rightarrow \infty}, \quad (2.15)$$

$$M(\partial/\partial t) = -2 N L K_{,r}|_{r \rightarrow \infty} - \frac{L}{N} K^3 W W_{,r}|_{r \rightarrow \infty} - 2\pi N K|_{r \rightarrow \infty} \epsilon_0, \quad (2.16)$$

where ϵ_0 stands for the energy density of the basis metric at infinity, which commonly is the AdS metric with M parameter. Incidentally, other useful representation of the mass is

$$M(r) = N(r)E(r) - W(r)J(r). \quad (2.17)$$

The evaluation of the mass for the studied charged dilaton solution yields

$$\begin{aligned} M(r, \epsilon_0) &:= -2 C_N r^{1/2 B k^2} L^2 - 2\pi r^{1+1/2 B k^2} L \epsilon_0, \\ M(r, \epsilon_0 = 0) &= -2 C_N C_1 - 8 \frac{r^{(4-Bk^2)/2} C_N \Lambda}{4 - Bk^2} - 8 \frac{r^{-1/2 B k^2} C_N Q^2}{Bk^2} \end{aligned} \quad (2.18)$$

Comparing with the energy characteristics of the BTZ solution, see Appendix A, one concludes that role of the mass parameter is played by $C_1 C_N = -M$. Recall that C_N can be equated to 1. At spatial infinity all physical quasi–local quantities occur to be infinite, in particular, the field Ψ logarithmically increases.

B. Algebraic classification of the field, energy–momentum, and Cotton tensors

The Maxwell field tensor is given by

$$(F^\mu{}_\nu) = \begin{bmatrix} 0 & -\frac{Q}{C_N L^2} r^{-1-1/2 B k^2} & 0 \\ -r^{-1+1/2 B k^2} L^2 Q C_N & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (2.19)$$

and is algebraically characterized by the following eigenvectors:

$$\begin{aligned} \lambda_1 &= r^{-1-Bk^2} Q : \mathbf{V1}, \mathbf{2} = [V^1, -r^{1/2 B k^2} L^2 C_N V^1, 0], V_\mu V^\mu = 0, \mathbf{V1} = \mathbf{N1}, \\ \lambda_2 &= -r^{-1-Bk^2} Q : \mathbf{V1} = [V^1, r^{1/2 B k^2} L^2 C_N V^1, 0], V_\mu V^\mu = 0, \mathbf{V2} = \mathbf{N2}, \\ \lambda_3 &= 0 : \mathbf{V3} = [0, 0, V^3], V_\mu V^\mu = V^{32} r^2, \mathbf{V3} = \mathbf{S3}, \end{aligned} \quad (2.20)$$

therefore, its type is $\{N, N, S\}$. For this class of charged dilatons, the energy-momentum tensor is described by

$$T^\mu{}_\nu = 2Q^2 r^{-2(1+Bk^2)} \delta^\mu{}_\phi \delta^\phi{}_\nu, \quad (2.21)$$

and is algebraically characterized by the following eigenvectors:

$$\begin{aligned} \lambda_{1,2} = 0 : \mathbf{V1}, \mathbf{2} &= [V1, V2, 0], \mathcal{N} := V_\mu V^\mu = -V^{12} C_N^2 r^{Bk^2} L(r)^2 + \frac{V^2{}^2}{L^2} \\ &, \mathbf{V1}, \mathbf{2} = \mathbf{T1}, \mathbf{2}, \mathbf{N1}, \mathbf{2}, \mathbf{S1}, \mathbf{2}, \\ \lambda_3 = 2Q^2 r^{-2-2Bk^2} : \mathbf{V3} &= [0, 0, V^3], V_\mu V^\mu = V^{32} r^2, \mathbf{V3} = \mathbf{S3}. \end{aligned} \quad (2.22)$$

Hence depending on the sign of the norm \mathcal{N} one will have spacelike, $\mathcal{N} > 0$, null, $\mathcal{N} = 0$, or timelike, $\mathcal{N} < 0$, eigenvectors: $\mathbf{V1}, \mathbf{2} = \{\mathbf{S}, \mathbf{N}, \mathbf{T}\}$. Therefore, in the case of a double root one may choose different vector components determining, for instance, one spacelike vector S and the other timelike T or null N one. Therefore, for the Maxwell energy tensor one may have the algebraic types: $\{2S, S\}, \{2N, S\}, \{2T, S\}$ and $\{(S, T), S\}, \{(T, T), S\}, \dots, \{(T, N), S\}$.

The Cotton tensor amounts to

$$\begin{aligned} C^\mu{}_\nu &= C^1{}_3 \delta^\mu{}_t \delta^\phi{}_\nu + C^3{}_1 \delta^\mu{}_\phi \delta^t{}_\nu, \\ C^3{}_1 &= -\frac{C_N}{32 r^4 B k^2} \left(r^{-Bk^2/2} C_1^2 B^2 k^4 (4 - Bk^2) + 12 r^{-Bk^2} Q^2 C_1 B k^2 (4 + Bk^2) \right. \\ &\quad \left. + 4 B^2 r^{2-Bk^2} \Lambda C_1 k^4 + 64 r^{-3Bk^2/2} Q^4 (2 + Bk^2) + 64 B r^{2-3Bk^2/2} \Lambda Q^2 k^2 \frac{(2 + Bk^2)}{(4 - Bk^2)} \right), \\ C^1{}_3 &= \frac{1}{32 C_N r^2} \left(B C_1 k^2 (4 - Bk^2) r^{-Bk^2} + 16 Q^2 (2 + Bk^2) r^{-3/2 Bk^2} \right). \end{aligned} \quad (2.23)$$

Depending on the signs of C_1 and $\Lambda = \pm 1/l^2$ the components of $C^1{}_3$ and $C^3{}_1$ may be positive or negative ones, consequently there product could be positive. Therefore, the Cotton tensor, for both $C^1{}_3$ and $C^3{}_1$ positive or negative, is characterized algebraically by

$$\begin{aligned} \lambda_1 = 0 : \mathbf{V1} &= [0, V^2, 0], V_\mu V^\mu = \frac{V^2{}^2}{L^2}, \mathbf{V1} = \mathbf{S1}, N^2 = C_N^2 r^{Bk^2} L^2, \\ \lambda_2 = \sqrt{C^1{}_3 C^3{}_1} : \mathbf{V2} &= [V^1, 0, V^1 \left(\frac{C^3{}_1}{C^1{}_3} \right)^{1/2}], \mathcal{N} = V_\mu V^\mu = -V^{12} (N^2 C^1{}_3 - r^2 C^3{}_1) / C^1{}_3, \\ \mathbf{V2} &= \mathbf{T2}, \mathbf{N2}, \mathbf{S2}, \\ \lambda_2 = -\sqrt{C^1{}_3 C^3{}_1} : \mathbf{V3} &= [V^1, 0, -V^1 \left(\frac{C^3{}_1}{C^1{}_3} \right)^{1/2}], V_\mu V^\mu = -V^{12} (N^2 C^1{}_3 - r^2 C^3{}_1) / C^1{}_3, \\ \mathbf{V3} &= \mathbf{T3}, \mathbf{N3}, \mathbf{S3}. \end{aligned} \quad (2.24)$$

Depending on the sign of the norm \mathcal{N} one has spacelike, $\mathcal{N} > 0$, null, $\mathcal{N} = 0$, or timelike, $\mathcal{N} < 0$, eigenvectors, denoted by: $\mathbf{V1}, \mathbf{2} = \{\mathbf{S}, \mathbf{N}, \mathbf{T}\}$. Therefore, when the eigenvalues are real, the Cotton tensor allows for types: $\{2S, S\}, \{2N, S\}, \{2T, S\}$ and $\{(S, T), S\}, \{(T, T), S\}, \dots, \{(T, N), S\}$.

In the spacetime region where $C^1{}_3$ and $C^3{}_1$ are of opposite signs one of the eigenvalues becomes imaginary and the following scheme arises

$$\begin{aligned} \lambda_1 = 0 : \mathbf{V1} &= [0, V^2, 0], V_\mu V^\mu = \frac{V^2{}^2}{L^2}, \mathbf{V1} = \mathbf{S1}, \\ \lambda_2 = \bar{\lambda}_3 = \sqrt{C^1{}_3 C^3{}_1} : \mathbf{V2} &= \mathbf{Z} = [V^1, 0, \frac{\sqrt{C^3{}_1}}{\sqrt{C^1{}_3}} V^1], \mathbf{V3} = \bar{\mathbf{Z}}, \end{aligned} \quad (2.25)$$

thus, the algebraic type of the Cotton tensor is $\{S, Z, \bar{Z}\}$.

III. STATIONARY CYCLIC SYMMETRIC BLACK HOLE SOLUTIONS COUPLED TO A SCALAR FIELD GENERATED VIA $SL(2, R)$ -TRANSFORMATIONS

Subjecting the metric (2.9) to a general $SL(2, R)$ -transformation

$$\begin{aligned} t &= \alpha T + \beta \Phi, \\ \phi &= \gamma T + \delta \Phi, \quad \Delta := \alpha \delta - \beta \gamma. \end{aligned} \tag{3.1}$$

one arrives at the stationary metric equipped with rotation

$$\begin{aligned} g &= - \left(\alpha^2 C_N^2 r^{Bk^2} L^2 - \gamma^2 r^2 \right) dT^2 + \frac{dr^2}{L^2} \\ &\quad - 2 \left(\beta \alpha C_N^2 r^{Bk^2} L^2 - \delta \gamma r^2 \right) dT d\Phi + \left(\delta^2 r^2 - \beta^2 C_N^2 r^{Bk^2} L^2 \right) d\Phi^2, \\ L(r)^2 &= \left(r^{Bk^2/2} C_1 + 4 \frac{r^2 \Lambda}{4 - Bk^2} + 4 \frac{Q^2}{Bk^2} \right) r^{-Bk^2}. \end{aligned} \tag{3.2}$$

Using the standard notation, this charged rotating dilaton solution is given as

$$\begin{aligned} g &= -N(r)^2 dT^2 + \frac{dr^2}{L(r)^2} + K(r)^2 (d\Phi + W(r) dT)^2, \\ N(r)^2 &= \frac{N^2 r^2}{\delta^2 r^2 - \beta^2 N^2} (\alpha \delta - \beta \gamma)^2, \\ N^2 &:= C_N^2 \left(r^{Bk^2/2} C_1 + 4 \frac{r^2 \Lambda}{4 - Bk^2} + 4 \frac{Q^2}{Bk^2} \right) = C_N^2 r^{Bk^2} L(r)^2, \\ L(r)^2 &= \left(r^{Bk^2/2} C_1 + 4 \frac{r^2 \Lambda}{4 - Bk^2} + 4 \frac{Q^2}{Bk^2} \right) r^{-Bk^2}, \\ K(r)^2 &= \delta^2 r^2 - \beta^2 N^2, \\ W(r) &= \frac{\delta \gamma r^2 - \beta \alpha N^2}{\delta^2 r^2 - \beta^2 N^2}, \quad \beta \neq 0 \neq \delta, \\ \Psi &= k \ln(r), \\ F_{\mu\nu} &= 2F_{Tr} \delta_{[\mu}^T \delta_{\nu]}^r + 2F_{\Phi r} \delta_{[\mu}^\Phi \delta_{\nu]}^r, \\ F_{Tr} &= -A_{T,r} = \alpha F_{tr} = -\alpha \frac{d}{dr} \left(2 \frac{Q C_N}{B k^2} r^{-Bk^2/2} \right), \quad F_{tr} = Q C_N r^{-1/2 Bk^2-1}, \\ F_{\Phi r} &= -A_{\Phi,r} = \beta F_{tr} = -\beta \frac{d}{dr} \left(2 \frac{Q C_N}{B k^2} r^{-Bk^2/2} \right). \end{aligned} \tag{3.3}$$

$$F_{tr} = Q C_N r^{-1/2 Bk^2-1} = -A_{t,r} \rightarrow A_t = 2 \frac{Q C_N}{B k^2} r^{-Bk^2/2},$$

It is worthwhile to notice that the structure of the electromagnetic energy tensor amounts to

$$T_{\mu\nu} = 2 Q^2 r^{-2 B k^2} \begin{bmatrix} \gamma^2 & 0 & \gamma \delta \\ 0 & 0 & 0 \\ \gamma \delta & 0 & \delta^2 \end{bmatrix}. \quad (3.4)$$

For the AdS ($\Lambda = 1/l^2$)–black hole branch; the constants appearing in the structural functions will be replaced by $C_N \rightarrow \sqrt{1 - 2 k^2} = B$, $k = \sqrt{1 - B^2}/\sqrt{2}$, $0 < k < 1/\sqrt{2}$, $1 > B > 0$, thus

$$N(r)^2 = N_B^2 = r^2 \left(r^{-2 B^2} C_1 B^2 + \Lambda \right), \quad L(r)^2 = L_B^2 = \frac{N_B^2}{B^2} r^{-4 + 4 B^2}. \quad (3.5)$$

A. Quasi local mass and momentum

The evaluation of the quasi-local momentum $J(r)$ yields

$$J(r) = J(r \rightarrow \infty) = \frac{1}{2} \beta \delta C_N C_1 (4 - B k^2) + 8 \frac{\delta C_N \beta Q^2}{B k^2} r^{-B k^2/2}, \quad (3.6)$$

hence, for positive $B > 0$, the contribution of the electromagnetic field $Q \neq 0$ to the momentum, at spatial infinity, disappears and one has

$$J(r \rightarrow \infty) = \frac{1}{2} \beta \delta C_N C_1 (4 - B k^2) =: J_0. \quad (3.7)$$

The evaluation of the quasi local mass yields

$$\begin{aligned} M(r) &= \frac{1}{2} \frac{C_N \Delta}{r^3 B k^2 (B k^2 - 4)} \frac{M_L}{D} - \frac{1}{2} \frac{\beta \delta C_N}{B^2 k^4 r^2} \frac{M_J}{D}, \\ M_L &:= -4 B k^2 C_1 r^3 (B k^2 - 4) (P + \Lambda \beta^2 C_N^2 B k^2) \\ &\quad + 4 r Q^2 \beta^2 C_N^2 C_1 (B k^2 - 4)^2 B k^2 + 16 B k^2 r^{5-1/2 B k^2} \Lambda P \\ &\quad + r^{1+1/2 B k^2} C_1^2 \beta^2 C_N^2 B^2 k^4 (B k^2 - 4)^2 - 16 Q^2 r^{3-1/2 B k^2} (B k^2 - 4) P, \\ M_J &= \left(16 B k^2 Q^2 r^{-B k^2/2} - B^2 C_1 k^4 \right) (\delta \gamma (B k^2 - 4) + 4 \Lambda C_N^2 \alpha \beta) (B k^2 - 4) r^2 \\ &\quad - 64 \beta \alpha r^{-1/2 B k^2} Q^4 C_N^2 (B k^2 - 4) + \beta \alpha B^2 k^4 C_1^2 C_N^2 r^{1/2 B k^2} (B k^2 - 4)^2 \\ &\quad + 4 \beta \alpha B k^2 C_1 C_N^2 Q^2 (B k^2 - 4) (B k^2 - 8), \\ D &= P - 4 \frac{\beta^2 C_N^2 Q^2 (B k^2 - 4)}{B r^2 k^2} - \beta^2 C_N^2 r^{(B k^2 - 4)/2} C_1 (B k^2 - 4), \end{aligned} \quad (3.8)$$

where $P = \delta^2 B k^2 - 4 \delta^2 + 4 \beta^2 C_N^2 \Lambda$. The order zero in the series expansion of $M(r)$ is given by

$$M_0 = 2 M \Delta \frac{\Lambda \beta^2 C_N^2 B k^2 + \delta^2 B k^2 - 4 \delta^2}{\delta^2 B k^2 - 4 \delta^2 + 4 \beta^2 C_N^2 \Lambda} - J_0 \frac{B k^2 \delta \gamma + 4 \beta \alpha C_N^2 \Lambda - 4 \delta \gamma}{\delta^2 B k^2 - 4 \delta^2 + 4 \beta^2 C_N^2 \Lambda}, \quad (3.9)$$

where $C_N C1$ has been replaced by $-M$. Comparing with the quasi-local BTZ mass $M_{BTZ} = 2M - 2r^2/l^2$, see details in Appendix A, when the rotation vanishes, $\beta = 0$, $\Delta = \alpha\delta \rightarrow 1$, implies that $M_0 \rightarrow -C_N C1 = M$.

It should be pointed out that, due to the presence of terms with positive powers of r in the series expansion of $M(r)$, it increases as $r \rightarrow \infty$, like the M_{BTZ} does.

Frequently, in the literature one encounters the $SL(2, R)$ transformation

$$t = \frac{T}{\sqrt{1 - \frac{\omega^2}{l^2}}} - \omega \frac{\Phi}{\sqrt{1 - \frac{\omega^2}{l^2}}}, \quad \phi = -\frac{\omega}{l^2} \frac{T}{\sqrt{1 - \frac{\omega^2}{l^2}}} + \frac{\Phi}{\sqrt{1 - \frac{\omega^2}{l^2}}}, \quad (3.10)$$

correspondingly, the structural functions (3.3) assume the form

$$\begin{aligned} \mathbf{N}(r)^2 &= \frac{(l^2 - \omega^2) r^2 N^2}{l^2 (r^2 - \omega^2 N^2)}, \quad \mathbf{L}(r) = \mathbf{L}(r), \quad N = N, \\ \mathbf{W}(r) &= \frac{\omega (N^2 l^2 - r^2)}{l^2 (r^2 - \omega^2 N^2)}, \quad \mathbf{K}(r)^2 = \frac{l^2 (r^2 - \omega^2 N^2)}{l^2 - \omega^2}. \end{aligned} \quad (3.11)$$

In this parametrization, the quasi-local momentum becomes $J_0 = -\frac{1}{2} \frac{\omega l^2}{l^2 - \omega^2} C_1 C_N (4 - B k^2)$, and vanishes as soon the rotation ω becomes zero.

The generated stationary metric (3.3) gives rise, among others, to a charged generalization of Chan–Mann rotating dilaton solution, see below Sec. III C.

With this result we are giving an answer to the remark contained in the Conclusions of [7]: “Although the static charged black solutions of (1.1) exist [6], at present we are unable to generalize our spinning solution to charged cases. This endeavor is complicated by the fact that when one adds Maxwell fields to a spinning solution, both electric and magnetic fields must be present...”

B. Algebraic classification of the field, energy–momentum, and Cotton tensors

The Maxwell electromagnetic field tensor amounts to

$$(F^\mu{}_\nu) = \begin{bmatrix} 0 & -\frac{\delta r^{-1-3/2} B k^2 Q}{\Delta C_N L^2} & 0 \\ -\alpha L^2 Q r^{-1-1/2} B k^2 C_N & 0 & -\beta L^2 Q r^{-1-1/2} B k^2 C_N \\ 0 & \frac{\gamma r^{-1-3/2} B k^2 Q}{\Delta C_N L^2} & 0 \end{bmatrix}, \quad (3.12)$$

and has the following eigenvectors

$$\begin{aligned} \lambda_1 = 0 : \mathbf{V1} &= [-\frac{\beta V^3}{\alpha}, 0, V^3], \quad V_\mu V^\mu = \frac{V^3 r^2 \Delta^2}{\alpha^2}, \\ \lambda_2 = Q r^{-1-Bk^2} : \mathbf{V2} &= [-r^{-Bk^2/2} \frac{\delta V^2}{\Delta L^2 C_N}, V^2, r^{-Bk^2/2} \frac{\gamma V^2}{\Delta L^2 C_N}], \quad V_\mu V^\mu = 0, \quad \mathbf{V2} = \mathbf{N2}, \\ \lambda_3 = -Q r^{-1-Bk^2} : \mathbf{V3} &= [r^{-Bk^2/2} \frac{\delta V^2}{\Delta L^2 C_N}, V^2, r^{-Bk^2/2} \frac{\gamma V^2}{\Delta L^2 C_N}], \quad V_\mu V^\mu = 0, \quad \mathbf{V3} = \mathbf{N3}. \end{aligned} \quad (3.13)$$

Correspondingly, its type is $\{S, N, N\}$.

The electromagnetic energy-momentum tensor is given by

$$(T^\mu{}_\nu) = -2 \frac{Q^2 r^{-(2+2Bk^2)}}{\Delta} \begin{bmatrix} \beta\gamma & 0 & \beta\delta \\ 0 & 0 & 0 \\ -\alpha\gamma & 0 & -\alpha\delta \end{bmatrix}, \quad (3.14)$$

and is algebraically characterized by:

$$\begin{aligned} \lambda_{1,2} = 0 : \mathbf{V1}, \mathbf{2} &= [V^1, V^2, -V^1\gamma/\delta], \mathcal{N} = V_\mu V^\mu = -\frac{C_N^2 L^2 r^{Bk^2} \Delta^2}{\delta^2} V^{1^2} + \frac{V^{2^2}}{L^2}, \\ \mathcal{N} > 0, \mathbf{V1}, \mathbf{2} &= \mathbf{S1}, \mathbf{2}, \mathcal{N} = 0, \mathbf{V1}, \mathbf{2} = \mathbf{N1}, \mathbf{2}, \mathcal{N} < 0, \mathbf{V1}, \mathbf{2} = \mathbf{T1}, \\ \lambda_3 = 2 Q^2 r^{-(2+2Bk^2)} : \mathbf{V3} &= [-\frac{\beta V^3}{\alpha}, 0, V^3], \\ V_\mu V^\mu &= \frac{(V^3)^2 r^2 \Delta^2}{\alpha^2}, \mathbf{V3} = \mathbf{S3}, \end{aligned} \quad (3.15)$$

In the case of a double root, depending on the sign of the norm $V_\mu V^\mu$, one will have spacelike, $\mathcal{N} > 0$, null, $\mathcal{N} = 0$, or timelike, $\mathcal{N} < 0$, eigenvectors: $\mathbf{V1}, \mathbf{2} = \{\mathbf{S1}, \mathbf{2}, \mathbf{N1}, \mathbf{2}, \mathbf{T1}, \mathbf{2}\}$. Therefore, one may choose different vector components determining, for instance, one spacelike vector S and the other timelike T or null N one. Therefore, one may have the algebraic Ricci types: $\{2S, S\}, \{2N, S\}, \{2T, S\}$ and $\{(S, T), S\}, \{(T, T), S\}, \dots, \{(T, N), S\}$, where parenthesis is used to stand out the multiplicity of the root under consideration.

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The matrix of the Cotton tensor for the rotating charged solution is given by

$$(C^\mu{}_\nu) = \begin{bmatrix} \frac{-\alpha\beta C^3_1 + \gamma\delta C^1_3}{\Delta} & 0 & \frac{-\beta^2 C^3_1 + \delta^2 C^1_3}{\Delta} \\ 0 & 0 & 0 \\ -\frac{\alpha^2 C^3_1 + \gamma^2 C^1_3}{\Delta} & 0 & -\frac{\alpha\beta C^3_1 + \gamma\delta C^1_3}{\Delta} \end{bmatrix} \quad (3.16)$$

where

$$\begin{aligned} C^1_3 &= \frac{1}{32 C_N r^2} \left(B C_1 k^2 (4 - Bk^2) r^{-Bk^2} + 16 Q^2 (2 + Bk^2) r^{-3/2 Bk^2} \right), \\ C^3_1 &= -\frac{C_N}{32 r^4 Bk^2} \left(r^{-Bk^2/2} C_1^2 B^2 k^4 (4 - Bk^2) + 12 r^{-Bk^2} Q^2 C_1 Bk^2 (4 + Bk^2) \right. \\ &\quad \left. + 4 B^2 r^{2-Bk^2} \Lambda C_1 k^4 + 64 r^{-3 Bk^2/2} Q^4 (2 + Bk^2) + 64 B r^{2-3 Bk^2/2} \Lambda Q^2 k^2 \frac{(2 + Bk^2)}{(4 - Bk^2)} \right). \end{aligned} \quad (3.17)$$

Because of the presence of C_1 and $\Lambda = \pm 1/l^2$ in C^1_3 and C^3_1 , these tensor components may be positive or negative quantities in some spacetime regions. Therefore, in general, the Cotton tensor can be characterized algebraically by three real eigenvalues

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$$\begin{aligned}
\lambda_1 = 0 : \mathbf{V1} &= [0, V^2, 0], V_\mu V^\mu = \frac{V^2}{L^2}, \mathbf{V1} = \mathbf{S1}, \\
\lambda_2 = \sqrt{C^1_3 C^3_1} : \mathbf{V2} &= [V^1, 0, V^3], \\
V^3 &= \frac{V1 \left(-\alpha \beta C^3_1 + \gamma \delta C^1_3 - \sqrt{C^3_1 C^1_3} \Delta \right)}{\delta^2 C^1_3 - \beta^2 C^3_1}, \\
V_\mu V^\mu &= \frac{V1^2 \Delta^2 (C^3_1 r^2 - C^1_3 N^2) (\delta^2 C^1_3 + \beta^2 C^3_1 + 2 \delta \beta \sqrt{C^3_1 C^1_3})}{(\delta^2 C^1_3 - \beta^2 C^3_1)^2}, \\
\mathbf{V2} &= \mathbf{T2}, \mathbf{N2}, \mathbf{S2}, \\
\lambda_3 = -\sqrt{C^1_3 C^3_1} : \mathbf{V3} &= [V^1, 0, V^3] \\
V^3 &= \frac{V1 \left(-\alpha \beta C^3_1 + \gamma \delta C^1_3 + \sqrt{C^3_1 C^1_3} \Delta \right)}{\delta^2 C^1_3 - \beta^2 C^3_1}, \\
V_\mu V^\mu &= \frac{V1^2 \Delta^2 (C^3_1 r^2 - C^1_3 N^2) (\delta^2 C^1_3 + \beta^2 C^3_1 - 2 \delta \beta \sqrt{C^3_1 C^1_3})}{(\delta^2 C^1_3 - \beta^2 C^3_1)^2}, \\
\mathbf{V3} &= \mathbf{T3}, \mathbf{N3}, \mathbf{S3}.
\end{aligned} \tag{3.18}$$

In this case the algebraic types of the Cotton tensor are given by $\{T, T, S\}$, $\{T, S, S\}$, ..., $\{S, S, S\}$.

For one real and two complex conjugate eigenvalues, one has

$$\begin{aligned}
\lambda_1 = 0 : \mathbf{V1} &= [0, V^2, 0], V_\mu V^\mu = \frac{V^2}{L^2}, \mathbf{V1} = \mathbf{S1}, \\
\lambda_2 = \bar{\lambda}_3 = \sqrt{C^1_3 C^3_1} : \\
\mathbf{V2} = \mathbf{Z} &= [V^1, 0, V^3], V^3 = -\frac{V1 \left(-\alpha \beta C^3_1 + \gamma \delta C^1_3 - \sqrt{C^3_1 C^1_3} \Delta \right)}{\delta^2 C^1_3 - \beta^2 C^3_1}, \\
\mathbf{V3} &= \bar{\mathbf{Z}}.
\end{aligned} \tag{3.19}$$

Thus, the algebraic type of the Cotton tensor is given by $\{S, Z, \bar{Z}\}$.

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C. Particular stationary cyclic symmetric black hole solution coupled to a scalar field via $SL(2, R)$ -transformation

Requiring the fulfillment of the relationship

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$$\gamma \delta = -4 \frac{C_N^2 \Lambda}{(B k^2 - 4)} \alpha \beta \tag{3.20}$$

the term with power r^2 in W or equivalently in $g_{T\Phi}$ disappears. The generalized rotating charged Chan–Mann metric is given by

$$\begin{aligned}
g &= - \left(C_N^2 \alpha^2 L^2 r^{Bk^2} - 16 \beta^2 \frac{C_N^4 \alpha^2 \Lambda^2 r^2}{\delta^2 (Bk^2 - 4)^2} \right) dT^2 + \frac{dr^2}{L^2} \\
&\quad - 2\alpha \beta \frac{C_N^2}{Bk^2} \left(r^{Bk^2/2} C_1 Bk^2 + 4 Q^2 \right) dT d\Phi + \left(\delta^2 r^2 - \beta^2 C_N^2 r^{Bk^2} L^2 \right) d\Phi^2, \\
L^2 &= \left(r^{Bk^2/2} C_1 + 4 \frac{r^2 \Lambda}{4 - Bk^2} + 4 \frac{Q^2}{Bk^2} \right) r^{-Bk^2}, \quad \Psi = k \ln(r), \\
F_{\mu\nu} &= 2F_{Tr} \delta_{[\mu}^T \delta_{\nu]}^r + 2F_{\Phi r} \delta_{[\mu}^\Phi \delta_{\nu]}^r, \quad F_{Tr} = \alpha F_{tr}, \quad F_{\Phi r} = \beta F_{tr}, \quad F_{tr} = Q C_N r^{-1-Bk^2/2}.
\end{aligned} \tag{3.21}$$

Switching of the rotation $\beta = 0$, the presented above metric reduces to the Chan–Mann charged dilaton metric.

IV. CONCLUDING REMARKS

Using the Schwarzschild coordinate frame for a static cyclic symmetric metric in $2 + 1$ gravity coupled minimally to a dilaton logarithmically depending on the radial coordinate in the presence of a exponential potential and an electrical Maxwell field the general solution of the Einstein–Maxwell–dilaton equations is derived avoiding any ansatz. This static solution occurs to be equivalent to the Chan–Mann charged dilaton static solution. Via a general $SL(2, R)$ –transformation of the Killing coordinates, applied on the derived charged static cyclic symmetric metric, a family of stationary dilaton solutions has been generated; they are equipped with five relevant parameters interpretable as dilaton parameter, charge, momentum, cosmological constant, and mass for some values of them. A particular $SL(2, R)$ –transformation is identified, which gives raise to the charged generalization of the rotating Chan–Mann dilaton solution. At spatial infinity all these solutions do not allow for an AdS–dS limit, there structural functions increase indefinitely as the radial coordinate increases. There exists a horizon, structurally common to the full class of solutions, determining their black hole character for a range of the physical parameters. This families of solutions, the static and stationary ones, have been characterized by their quasi-local energy, mass, and momentum through their series expansions at spatial infinity. The algebraic classifications of the electromagnetic field, Maxwell energy–momentum, and Cotton tensors are established. The electromagnetic field tensor belongs to the type $\{S, N, N\}$ The Maxwell energy–momentum tensor is of types: $\{2S, S\}$, $\{2N, S\}$, $\{2T, S\}$, $\{(S, T), S\}$, $\{(T, T), S\}$, ..., $\{(T, N), S\}$. The Cotton tensor exhibits various possibilities. For real roots it falls into types: $\{T, T, S\}$, $\{T, S, S\}$, $\{T, N, S\}$, $\{S, T, S\}$, $\{S, N, S\}$, ..., $\{S, S, S\}$. For one complex root, the Cotton tensor is of the type $\{S, Z, \bar{Z}\}$.

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Appendix A: Momentum, energy and mass of the Bañados–Teitelboim–Zanelli black hole

Let us consider the asymptotically anti-de Sitter (2 + 1)–dimensional stationary black hole solution—the BTZ–metric— given by

$$ds^2 = -N(\rho)^2 dt^2 + \frac{1}{L(\rho)^2} d\rho^2 + \rho^2 [d\phi + W(\rho)dt]^2,$$

$$N^2(\rho) = L^2(\rho) = -M + \frac{\rho^2}{l^2} + \frac{J^2}{4\rho^2}, \quad K(\rho) = \rho, \quad W(\rho) = -\frac{J}{2\rho^2}. \quad (\text{A1})$$

The corresponding surface energy and momentum densities, at $\rho = R = \text{costant}$, are equal to

$$\epsilon(R, \epsilon_0) = -\frac{1}{\pi R} \sqrt{-M + \frac{R^2}{l^2} + \frac{J^2}{4R^2}} - \epsilon_0, \quad j_\phi(R) = \frac{1}{2\pi} \frac{J}{R}. \quad (\text{A2})$$

Consequently the total momentum, energy, and mass are

$$J(\partial/\partial\phi) = J,$$

$$E(R, \epsilon_0) = -2 \sqrt{-M + \frac{R^2}{l^2} + \frac{J^2}{4R^2}} - 2\pi\epsilon_0,$$

$$M(\partial/\partial t) = N(R) E(R, \epsilon_0) + \frac{J^2}{2R^2} = 2M - 2\frac{R^2}{l^2} - 2\pi\epsilon_0 \sqrt{-M + \frac{R^2}{l^2} + \frac{J^2}{4R^2}}. \quad (\text{A3})$$

These expressions for surface densities and global quantities are in full agreement with the corresponding ones reported in Ref. [12], section IV.

Notice that the series expansion of the energy and mass independent of ϵ_0 behave at infinity R , which will be denoted from now on by the same coordinate Greek letter ρ accompanied by $\rightarrow \infty$ and the approximation sign \approx , as

$$\begin{aligned} \epsilon(\rho \rightarrow \infty, \epsilon_0 = 0) &\approx -\frac{1}{\pi l} + \frac{l M}{2\pi \rho^2}, \quad E(\rho \rightarrow \infty, \epsilon_0 = 0) \approx -\frac{2\rho}{l} + \frac{l M}{\rho}, \\ M(\rho \rightarrow \infty, \epsilon_0 = 0) &\approx 2M - 2\frac{\rho^2}{l^2}. \end{aligned} \quad (\text{A4})$$

Although the expression of $M(\rho, \epsilon_0 = 0)$ holds in the whole spacetime and not only in the boundary at spatial infinity, the approximation sign \approx is used instead of the equality $=$ to be consistent with the point under consideration. The reference energy density to be used in this work is the one corresponding to the anti-de Sitter metric with parameter M_0 , $\epsilon_0(M_0) = -\frac{1}{\pi\rho}\sqrt{\frac{\rho^2}{l^2} - M_0}$, $\epsilon_{0|\infty}(M_0) \approx -\frac{1}{\pi l} + \frac{l M_0}{2\pi \rho^2}$, then the expansions of the physical characteristics at spatial infinity, $\rho \rightarrow \infty$, are given as

$$\begin{aligned} \epsilon(\rho \rightarrow \infty, \epsilon_{0|\infty}(M_0)) &\approx \frac{l}{2\pi \rho^2}(M - M_0), \quad E(\rho \rightarrow \infty, \epsilon_{0|\infty}(M_0)) \approx l \frac{(M - M_0)}{\rho}, \\ M(\rho \rightarrow \infty, \epsilon_{0|\infty}(M_0)) &\approx M - M_0. \end{aligned} \quad (\text{A5})$$

Thus, comparing the quasi local energy and mass with the corresponding quantities associated to the AdS solution one sees that energies vanish while mass stays finite as the radial coordinate approaches infinity. For the proper AdS reference metric one has to equate $M_0 = -1$.